AIAA 81-4010

New, Quasi-simultaneous Method to Calculate Interacting Boundary Layers

A. E. P. Veldman*
National Aerospace Laboratory NLR, Amsterdam, The Netherlands

A quasi-simultaneous method is described to calculate laminar, incompressible boundary layers interacting with an inviscid outer flow. The essential feature of this method is an interactive boundary condition prescribing a linear combination of pressure and displacement thickness which models the behavior of the outer flow. This way the quasi-simultaneous method avoids difficulties incurred when either direct or inverse methods are used, resulting in fast convergence of the iterative procedure involved. The method is consistent with the asymptotic triple-deck theory. Results will be presented for two problems which exhibit strong interaction between the viscous and inviscid regions: a boundary layer with a separation bubble, and the flow near the trailing edge of a flat plate.

Nomenclature

c_f	= skin friction coefficient, $c_f = Re^{-1}\partial u/\partial y$
h	= mesh size in x direction
H	= thickness of computational boundary-layer domain
L	= characteristic length
M,N	= number of grid points in y and x direction, respectively
p	= pressure
R_i	= remainder term in discrete interaction law
Re	= Reynolds number, $Re = U_0 L/\nu$
u	= component of flow velocity parallel to surface
u_e, u_{e_0}	= value of u at edge of boundary layer with and without interaction, respectively
U_0	= characteristic velocity
x	= coordinate along surface
x_b, x_e	= endpoints of region of interaction
$y, \tilde{y}$	= coordinates normal to surface, $\tilde{y} = Re^{1/2} y$
α_{ij}	= matrix coefficients in discrete interaction law
δ^*, δ	= displacement thickness, $\delta = Re^{1/2}\delta^*$
ν	= kinematic viscosity
$\psi, \tilde{\psi}$	= stream functions, $\tilde{\psi} = Re^{1/2}\psi$
ω	= relaxation factor

Subscripts

i,j	= grid point location

Superscript

n	= number of global iteration sweep

I. Introduction

IT is well-known that many numerical methods for solving the boundary-layer equations encounter difficulties when the flow tends to separate from the body surface. The relevant common feature of these methods is the prescription of the pressure as a boundary condition (the direct method). It has been shown by Goldstein that, apart from exceptional circumstances, this will inevitably lead to a singularity in the solution at the point of separation,[1,2] causing the breakdown of the numerical method. As an immediate consequence the conventional direct iterative schemes which are being used to account for the interaction between the boundary layer and the outer inviscid flow cannot be applied when separation is present.

Catherall and Mangler[3] were the first to realize that relaxing the pressure might lead to regular solutions of the boundary-layer equations. Indeed, prescribing the displacement thickness as a boundary condition, they were able to integrate the boundary-layer equations through the separation point and into a region of reverse flow without any evidence of singular behavior at the separation point. This observation has subsequently led to other inverse solutions of the boundary-layer equations, prescribing either displacement thickness or wall shear.[4-7]

A problem associated with the latter, so-called inverse, techniques is that the required displacement thickness (or wall shear) is not known a priori. The appropriate value has to be obtained, as part of the overall problem, from the interaction between the boundary layer and the inviscid flow. Calculations of subsonic and transonic flows which are essentially based upon this inverse type of viscous-inviscid interaction have been presented by several authors.[8-12]

The global organization of the direct and inverse iterative methods to calculate the interaction problem has been sketched in Fig. 1. The calculation consists of two parts treated alternately: the calculation of the boundary layer (viscous region), and the calculation of the outer flow (inviscid region). In the direct method the boundary-layer equations are solved with prescribed pressure p, and the outer flow is computed with prescribed displacement thickness δ^*. In the inverse method the role of p and δ^* is interchanged.

As discussed above, the direct method breaks down as soon as boundary-layer separation occurs. The inverse method can treat this situation, but the convergence of the iterative process is slow due to the need for severe underrelaxation. For instance, in Ref. 10 a relaxation factor of 0.2 had to be used. The convergence gets even slower when the region of inverse boundary-layer calculation is enlarged. A second drawback is the difficulty to obtain a smooth transition from a directly to an inversely calculated region.[8]

An improvement over the inverse method can be obtained by applying the semi-inverse method introduced by Le Balleur.[9] Herein the boundary layer is calculated inversely, but the outer flow is calculated in the direct way. Thus both parts of the flowfield are computed with prescribed displacement thickness, resulting in two pressure distributions which must be the same upon convergence. An iterative cycle is completed with a relaxation formula which combines the old displacement thickness and the two pressure distributions into an updated displacement thickness. The organization of this method has also been sketched in Fig. 1. The method of Kwon and Pletcher[11] and Carter[12] possesses the same type of organization, but the relaxation formula is different from the one used by Le Balleur.[9] The choice for the relaxation formula made by Kwon and Pletcher[11] and Carter[12] is rather

Received July 2, 1980.

*Research Scientist.

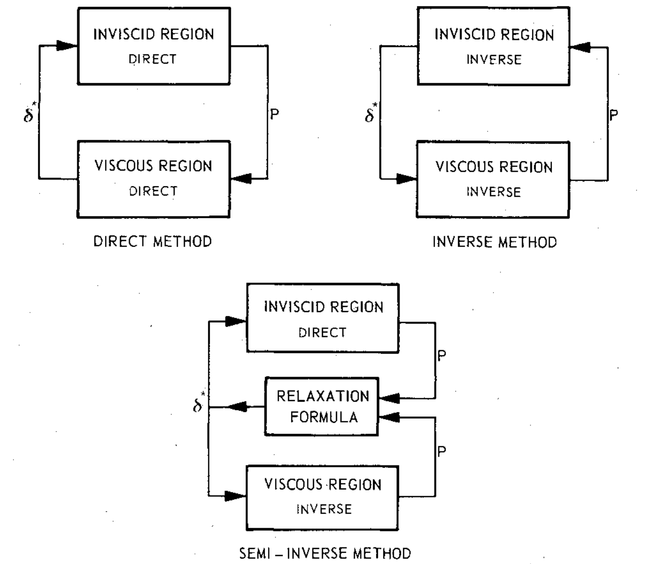

Fig. 1 Global organization of direct, inverse, and semi-inverse methods.

satisfactory as the iterative process allows overrelaxation leading to fast convergence, whereas the method of Le Balleur[9] still requires underrelaxation. However, the process of choosing the relaxation formula still contains some arbitrariness, since, as reported by Carter,[12] other plausible choices for the relaxation formula can lead to divergent iterative schemes; hence extension of the method to more general situations is not straightforward.

In the methods mentioned above with the alternate treatment of viscous and inviscid regions, the numerical scheme provides only a weak, retarded coupling of both regions. However, near separation and also near trailing edges, the interaction has a strong, simultaneous character, i.e., there does not exist a definite hierarchy between the boundary layer and the outer flow.[13,14] As a consequence of this observation it may be expected that a numerical scheme which also possesses a simultaneous coupling between the viscous and inviscid parts of the flowfield can have favorable properties. The iterative methods used for many years in case of a supersonic outer flow do implement this simultaneous character already[15,16] by applying a tangent wedge condition or a Prandtl-Meyer relation as a boundary condition to the boundary-layer equations.

In the present paper we will introduce a subsonic analog of the latter methods. The boundary-layer equations will be solved with a suitable representation of the interaction law as a boundary condition, which in this case possesses the elliptic character of the outer flow. This way the method prescribes a linear combination of pressure and displacement thickness, in contrast with the methods mentioned above which prescribe either one or the other. The iterative process, required because of the elliptic boundary condition, appears to converge rapidly; overrelaxation (with a factor 1.5 typically) can be applied. Due to a close resemblance with direct methods, smooth transition to directly calculated regions can easily be achieved.

The new method is consistent with the asymptotic triple-deck theory of Stewartson.[17] This theory describes, in the limit of vanishing viscosity, the structure of a laminar boundary layer in the vicinity of singular points such as a trailing edge or a point of separation. The present method can be regarded as a translation of the asymptotic theory for $Re \to \infty$ into a model for large values of Re. Therefore the paper is begun with a review of the essential features of the triple-deck. Hereafter the new calculation method will be presented. Special attention will be given to the essential numerical features. The method is demonstrated on two problems, covering two different situations in which strong interaction occurs: a boundary layer with a separation bubble, introduced by Carter and Wornom[10] and the flow near the trailing edge of a flat plate. In the latter case the results of the present interacting boundary-layer calculations are compared with the results from triple-deck theory.

II. Description of (Strongly) Interacting Boundary Layers

Asymptotic Point of View

The nature of the interaction between viscous and inviscid regions can be studied by means of the asymptotic theory for $Re \to \infty$. The asymptotic theory indicates which set of equations is required to describe the flow. Moreover it can suggest how to choose the numerical method for solving the flow equations. Situations in which the conventional direct boundary-layer solutions, i.e., with pressure prescribed, behave smoothly can be described by the classical boundary-layer theory as originated by Prandtl. However, when the direct solution exhibits singularities, a more refined theory is required. In case of laminar flow a number of these singular cases can be treated by the so-called triple-deck theory introduced by Stewartson and Messiter. A comprehensive review hereof has been given by Stewartson.[17] The applications of this theory include the flow near cusped trailing edges with finite curvature,[18] near separation,[19] and presumably near reattachment as well.[20]

The triple-deck is a three-layered region which in incompressible flow, to which we will restrict ourselves, has a streamwise extent $O(Re^{-3/8}L)$ around the singular point referred to in the above situations. In normal direction it consists of:

1) A viscous sublayer of thickness $O(Re^{-5/8}L)$ which is governed by the boundary-layer equations.

2) An inviscid middle layer of thickness $O(Re^{-1/2}L)$ where, effectively, the oncoming flow undergoes a downward displacement.

3) An inviscid irrotational top layer of thickness $O(Re^{-3/8}L)$ which can be described by the thin airfoil theory.

A sketch of the triple-deck is presented in Fig. 2.

The triple-deck possesses a number of properties, to be described below, which in our opinion are relevant for the design of a calculation method for interacting boundary layers:

1) The boundary-layer equations are sufficient to describe the lower two layers of the triple-deck, i.e., the viscous part of the flow. This property is consistent with the findings of Briley and McDonald[21] who have compared the solution of the boundary-layer equations with the solution of the Navier-Stokes equations for a boundary-layer flow with a separation bubble.

2) The interaction with the outer flow has a local character, as the triple-deck has only a small extent $O(Re^{-3/8}L)$. Furthermore the interaction can be described by the thin airfoil theory. It may therefore be expected that a local linearized description of the outer flow will suffice to give a good approximation of the interaction. For situations in which the outer flow is supersonic this property is exploited in interacting boundary-layer models where a linearized Prandtl-Meyer relation is being used to describe the interaction. One of the first practical examples hereof is due to Lees and Reeves.[16] A subsonic version of the latter method has been presented by Crimi and Reeves.[22]

3) There is no definite hierarchy between the viscous region and the inviscid region. In fact, far upstream and downstream in the triple-deck, i.e., $Re^{3/8}|x| \to \infty$ but $x \to 0$, the hierarchy of the flow tends to the classical direct boundary-layer hierarchy where the pressure is determined mainly by the outer inviscid region. When $Re^{3/8}x \to 0$ the hierarchy changes into the inverse type, in which the pressure distribution is

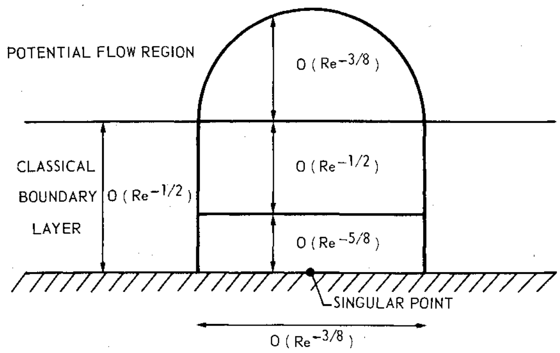

Fig. 2 Triple-deck.

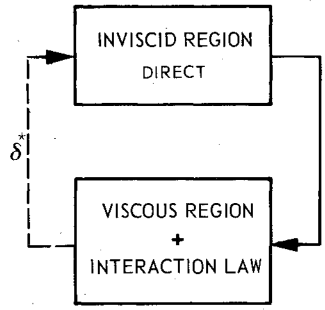

Fig. 3 Global organization of present, quasi-simultaneous, method.

determined in the boundary layer.[14] Thus in the triple-deck the hierarchy changes its character from direct into inverse. Such a situation in which the interaction does not possess a definite hierarchy will be defined *strong interaction.* The change in hierarchy in the triple-deck can be used to explain the convergence difficulties of the direct and inverse methods for calculating interacting boundary layers. For it is observed that the numerical difficulties originate where the hierarchy of the numerical scheme differs from the asymptotic hierarchy. A detailed study of this aspect has been published elsewhere.[23] This study also reveals how a numerical scheme without hierarchy can avoid these difficulties. Such a scheme will be presented in this paper.

The above asymptotic description of (strongly) interacting boundary layers is valid for laminar, incompressible flow. For supersonic flow,[17] turbulent flow,[24] and shock-wave boundary-layer interaction[25] the situation becomes more complicated. Nevertheless the same type of conclusions can be drawn from the appropriate asymptotic theory, and they will probably be of the same help as in the present case for the design of an efficient calculation method.

Interacting Boundary-Layer Model

We will now come to the presentation of the interacting boundary-layer model. Consistent with triple-deck property 1, it is assumed that the viscous part of the flow can be described by the boundary-layer equations:

$$uu_x - \tilde{\psi}_x u_{\tilde{y}} = u_e u_{e_x} + u_{\tilde{y}\tilde{y}} \tag{1a}$$

$$u = \tilde{\psi}_{\tilde{y}} \tag{1b}$$

Here $\tilde{y}$ is a scaled normal coordinate, $\tilde{y} = Re^{1/2}y$; (x,y) isa Cartesian coordinate system; $\tilde{\psi}$ is the scaled streamfunction, $\tilde{\psi} = Re^{1/2}\psi$; $u = \psi_y$ is the velocity component in x direction; and u_e denotes the value of u at the edge of the boundary layer.

The interaction with the outer flow will be expressed by means of a boundary condition for $\tilde{y} \rightarrow \infty$. We will formulate this interaction law in external velocity rather than pressure. Let u_{e_0} be the external velocity resulting from a first-order potential flow calculation, and let $u_{e_{\delta^*}}$ denote the correction hereof due to a displacement thickness δ^*. Then the interaction law can be written as

$$u_e(x) = u_{e_0}(x) + u_{e_{\delta^*}}(x) \tag{2}$$

In many situations it will be sufficient to use an approximate version of $u_{e_{\delta^*}}$. In supersonic flow one often uses a linearized Prandtl-Meyer relation.[16] For a subsonic flow a thin airfoil approximation may be used,[22] which is consistent with property 2 of the triple-deck. Thus we will use the following approximate interaction law

$$u_e(x) = u_{e_0}(x) + \frac{1}{\pi} Re^{-1/2} \oint_{x_b}^{x_e} \frac{\mathrm{d}(u_{e_0}\delta)/\mathrm{d}\xi}{x-\xi} \mathrm{d}\xi \tag{3}$$

where $[x_b, x_e]$ denotes the region in which the interaction is important. Further, δ is a scaled displacement thickness, $\delta = Re^{1/2}\delta^*$, defined by the requirement

$$\tilde{y} \rightarrow \infty: \tilde{\psi} \sim u_e(\tilde{y} - \delta) \tag{4}$$

The system formed by Eqs. (1), (3), and (4) is completed with initial conditions, and boundary conditions at $\tilde{y} = 0$.

The essential point of the present method is the numerical treatment of the interaction law [Eq. (3)]. In this relation both u_e and δ are (simultaneously) treated as unknowns. Thus a linear combination of u_e and δ is prescribed as a boundary condition, avoiding a numerical hierarchy as recommended by triple-deck property 3. More details about this approach will be given in Sec. III. Since the approximate interaction law, which describes the most important influence of the outer flow, is treated simultaneously with the boundary layer, the present method will be called quasi-simultaneous. The organization of this method is sketched in Fig. 3; this figure can be compared with Fig. 1. In cases where the approximation of the interaction law is not accurate enough it may be necessary to update the outer flow calculation. This possibility is indicated by the dashed line in Fig. 3.

III. Numerical Formulation

Discretization of the Boundary-Layer Equations

In the examples to be presented the boundary-layer calculation has been restricted to the same x interval $[x_b, x_e]$ where the interaction is believed to be important. The outer edge of the boundary layer has been placed at a finite distance, $\tilde{y} = H(x)$, from the body surface. A good choice appears to be $H(x) \doteq 7\delta(x)$. The bounded domain thus obtained has been covered by a (nonequidistant) finite-difference mesh with gridpoints $(x_i, \tilde{y}_j)$; $i = 0, ..., N$; $j = 0, ..., M$.

The x derivatives appearing in Eq. (1) have been replaced by the familiar three-point backward formula. The $\tilde{y}$ derivatives in Eq. (1a) have been discretized with second-order formulas centered around $\tilde{y}_j$, $j = 1, ..., M-1$; whereas in Eq. (1b) they have been centered around $\tilde{y}_{j+1/2}$, $j = 0, ..., M-1$. The resulting set of discrete equations at an x station x_i has been combined with the boundary conditions at $\tilde{y}_0 = 0$ and $\tilde{y} \rightarrow \infty$. The discretized version of the interactive condition [Eq. (3)] will be described in the following subsections. The complete set of nonlinear equations is solved with Newton's method. The resulting matrix is sparse, i.e., 2 × 2 block tridiagonal with one additional column, and is solved taking the sparsity pattern into account.

We stress that in the present calculations the three-point backward scheme has also been used in regions of backflow. According to Cebeci and Bradshaw[26] (p.370) all inverse boundary-layer procedures use the FLARE approximation for regions of backflow. This approximation has been introduced by Flügge-Lotz and Reyhner,[27] and it neglects the uu_x term in Eq. (1a) in the region with negative u component. The purpose of this approximation is to stabilize the numerical solution process. However, for the separated flow calculations performed with the present quasi-simultaneous method it has not appeared necessary to use this approach.

Discretization of the Interaction Law

In the calculations we have applied thus far a simple discretization of the thin airfoil integral in Eq. (3), namely the midpoint rule, appeared to give sufficiently accurate results. This has been verified by comparing the results with those obtained when using a much more refined treatment; details can be found in Ref. 28.

To describe the discretization of Eq. (3), let the x coordinates of the mesh points be given by $x_i = x_b + ih$, $i=0,\ldots,N$; $x_N = x_e$. Further let A_j be an abbreviation for $(u_{e_0}\delta)(x_j)$. The integral in Eq. (3) - to be calculated for $x=x_i$, $i=1,\ldots,N$ - will be evaluated with the midpoint rule on intervals $[x_j, x_{j+1}]$, $j=0,\ldots,N-1$. This results in

$$J_i \stackrel{\mathrm{df}}{=} \oint_{x_b}^{x_e} \frac{A'}{x_i-\xi} \mathrm{d}\xi = h \sum_{j=0}^{N-1} A'_{j+1/2} / (x_i - \xi_j - 1/2 h) \tag{5}$$

The value of $A'_{j+1/2}$ has been discretized as

$$A'_{j+1/2} = (A_{j+1} - A_j)/h$$

which after substitution into Eq. (5) leads to

$$hJ_i = \sum_{j=0}^{N-1} (A_{j+1} - A_j)/(i-j-1/2)$$

$$= -A_0/(i-1/2) + A_N/(i-N-1/2) - \sum_{j=1}^{N} A_j/[(i-j)^2 - 1/4]$$

where we have manipulated with A_N such that all A_i get the same coefficient in the summation over j. The last expression will be abbreviated to

$$hJ_i = R_i^* + \sum_{j=1}^{N} c_{ij} A_j \tag{6}$$

where

$$c_{ij} = -[(i-j)^2 - 1/4]^{-1}$$

$$R_i^* = -A_0/(i-1/2) + A_N/(i-N-1/2)$$

It can be deduced that the matrix (c_{ij}) is symmetric and diagonally dominant. Moreover $c_{ii} > 0$, hence (c_{ij}) is also positive definite. These properties have an important influence on the convergence of the global iteration process (to be described below). This is discussed in more detail in Ref. 23. Therefore, in case another discretization of the interaction integral has to be used one should strive for a positive definite, diagonally dominant matrix of coefficients c_{ij}.

Finally, substituting the discretization of Eq. (6) into Eq. (3) we obtain the discrete interaction law

$$u_{e_i} = u_{e_{0_i}} + \sum_{j=1}^{N} \alpha_{ij}\delta_j + R_i \tag{7}$$

where

$$\alpha_{ij} = \beta u_{e_{0_i}} c_{ij},\ R_i = \beta R_i^*,\ \beta = (\pi h Re^{1/2})^{-1}$$

Implementation of the Interaction Law

The essential feature of the present method is the way in which the interaction law is coupled numerically with the boundary layer equations. As already mentioned in Sec. II we treat both u_{e_i} and δ_i as unknowns when calculating the ith boundary-layer station. Due to the elliptic character of the interaction law several sweeps through the boundary layer have to be made. In the nth sweep the implementation of Eq. (7) is as follows

$$u_{e_i}^{(n)} - \alpha_{ii}\delta_i^{(n)} = u_{e_{0_i}} + \sum_{j=1}^{i-1} \alpha_{ij}\delta_j^{(n)} + \sum_{j=i+1}^{N} \alpha_{ij}\delta_j^{(n-1)} + R_i^{(n-1)} \tag{8}$$

To show the difference with the direct and inverse methods we will write below the corresponding versions of Eq. (8). The direct version reads

$$u_{e_i}^{(n)} = u_{e_{0_i}} + \sum_{j=1}^{N} \alpha_{ij}\delta_j^{(n-1)} + R_i^{(n-1)} \tag{9}$$

and the inverse version would be

$$-\alpha_{ii}\delta_i^{(n)} = u_{e_{0_i}} - u_{e_i}^{(n-1)} + \sum_{j=1}^{i-1} \alpha_{ij}\delta_j^{(n)} + \sum_{j=i+1}^{N} \alpha_{ij}\delta_j^{(n-1)} + R_i^{(n-1)} \tag{10}$$

although actually in most inverse methods a discrete version of the inverse thin airfoil integral will be used in the interaction law [Eq. (3)], resulting in a different right-hand side in Eq. (10).

The present implementation [Eq. (8)] of the interaction law leads to the prescription of a linear combination of u_e and δ as a boundary condition to the viscous flow equations. Numerical experiments with different types of the linear combination, say $u_e + c\delta$ with varying c, have revealed that not just any combination can be used. It is known that the case $c=0$ [corresponding with Eq. (9)] induces a breakdown of the boundary-layer calculation at a point of separation. The numerical experiments have shown that the choice $c>0$ leads to a similar breakdown (at another position). Negative choices for c, with c not too close to zero, do not give rise to numerical difficulties in the boundary-layer calculation. In particular, the value of c defined by Eq. (8) appears to be a favorable one. More details about this point can be found in Refs. 23 and 28.

The implementation of the interaction law not only affects the solution of the boundary-layer equations at a given station i, but it also affects the overall convergence of the global iteration process. For instance the inverse choice, $c = -\infty$, results in very poor convergence (only after sufficient underrelaxation), as discussed in the introduction. The present choice $c = -\alpha_{ii}$ appears to lead to much better convergence properties.

The Global Iteration Process

To complete the description of the numerical formulation, an overview of the organization of the global iteration process, required because of the elliptic nature of the outer flow, will be given below.

Step 0: $n=0$. Calculate u_{e_0} by means of the outer flow equations. Prescribe an initial profile at $x=x_b$, e.g., the Blasius profile. Give a first estimate $\delta^{(0)}$ for the displacement thickness distribution, e.g., the Blasius distribution.

Step 1: $n=n+1$. Update the thickness of the domain of computation, $H^{(n)} = 7\delta^{(n-1)}$. March through the boundary layer from x_1 to x_N in the way described in the preceeding subsections. The interaction law is implemented as indicated by Eq. (8). We note that the displacement thickness is treated in a Gauss-Seidel fashion.

Step 2: After a complete sweep through the boundary layer the iteration process is checked for convergence, where the convergence criterion has been taken as

$$\max_i |\delta_i^{(n)} - \delta_i^{(n-1)}| < 10^{-4} \tag{11}$$

If this criterion is not yet satisfied δ is overrelaxed with a factor $\omega = 1.5$ (typically), after which the process returns to step 1. It appears that the overall convergence is not very sensitive to the choice of ω.

Remark: An analogous approach has been successfully used to solve the triple-deck equations.[23] Again overrelaxation can be applied, whereas the usual inverse methods require underrelaxation with $\omega < 0.1$ (typically).

IV. Examples

The quasi-simultaneous method will be demonstrated by means of two examples of laminar, incompressible boundary layers which exhibit a strong interaction with the outer flow:

1) A boundary layer with a separation bubble, calculated earlier by Carter and Wornom.[10]

2) The viscous flow near the trailing edge of a flat plate (symmetrical).

Separation Bubbles

The present method will be demonstrated first by recalculating a problem defined by Carter and Wornom.[10] They have considered a dented plate given by

$$y_B = -0.03\text{sech}4(x-2.5), 0 \leq x < \infty$$

placed in an oncoming parallel flow with unit velocity (see Fig. 4). The Reynolds number has been chosen $Re = 1/\nu = 8 \times 10^4$. Carter and Wornom assume that the pressure distribution of the first-order outer potential flow past the body can be represented by the thin airfoil theory:

$$u_{e_0}(x) = 1 + \frac{1}{\pi} \oint_0^\infty \frac{\mathrm{d}y_B/\mathrm{d}\xi}{x-\xi} \mathrm{d}\xi$$

For purposes of comparison we have also used this assumption. Further it has been assumed that the interaction takes place between $x = 1$ and 4. Moreover, to be consistent with Carter and Wornom, the interaction formula (3) has been used with $u_{e_0}\delta$ replaced by δ, thus exploiting the fact that u_{e_0} does not differ too much from unity.

Carter and Wornom have applied an inverse method requiring pronounced underrelaxation ($\omega = 0.3$). For a 121 × 87 grid they need 64 iterations to pass the convergence test [Eq. (11)], which requires about 30 min computation time on a CDC 6600. With the quasi-simultaneous method, using overrelaxation with $\omega = 1.5$, about 10-20 iterations, depending on the mesh size, are sufficient to satisfy Eq. (11). The solution for a 121 × 81 grid can be calculated in less than 4 min on a CDC 6600.

Some results obtained with the present method have been plotted in Figs. 5-8. Figure 5 shows the displacement

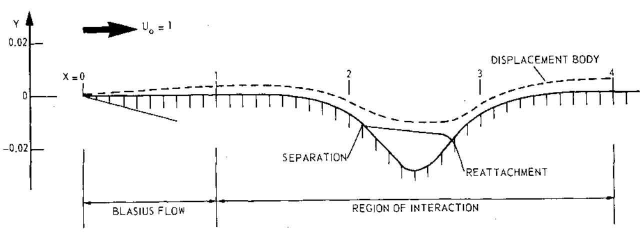

Fig. 4 Dented plate for calculation of separation bubbles (separation streamline and displacement body are indicated for $Re = 36 \times 10^4$). Note difference in x and y scale.

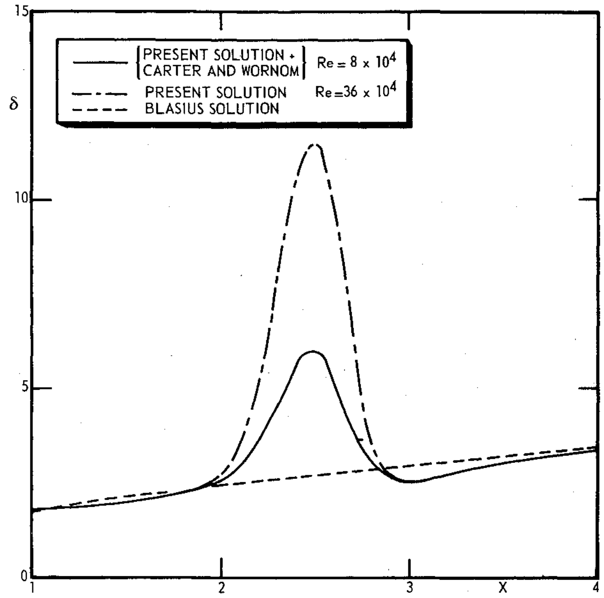

Fig. 5 Separation bubble: displacement thickness distributions.

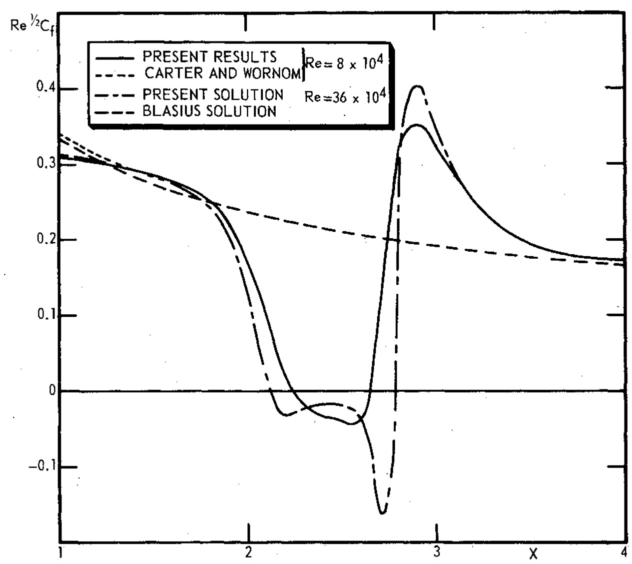

Fig. 6 Separation bubble: skin friction.

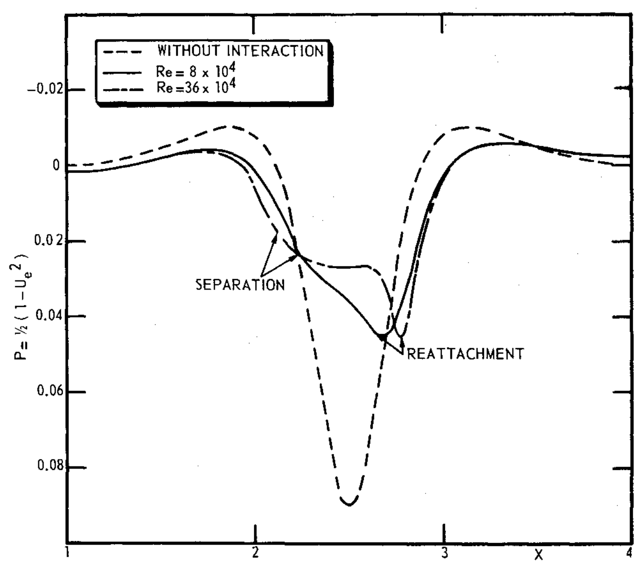

Fig. 7 Separation bubble: pressure distributions with and without separation.

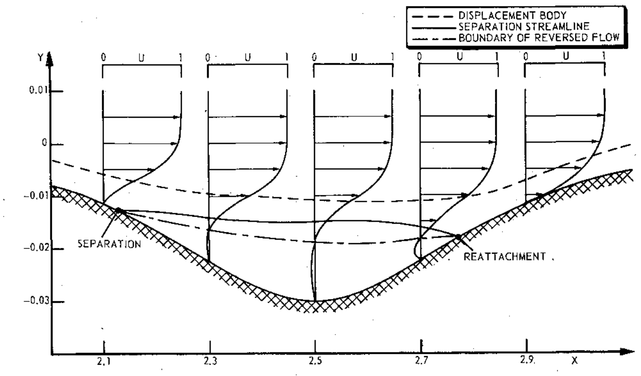

Fig. 8 Separation bubble: velocity profiles for case with severe separation, $Re = 36 \times 10^4$.

thickness. Besides our and Carter and Wornom's results for $Re = 8 \times 10^4$, which agree within graphical accuracy, this figure also shows results obtained with the present method for $Re = 36 \times 10^4$. Moreover, for comparison the displacement thickness for a flat plate (Blasius solution) has been inserted. The latter values have been used as the initial guess in the calculations. The skin friction distribution is displayed in Fig. 6. Here a difference between the present results for $Re = 8 \times 10^4$ and those of Carter and Wornom is visible near the upstream end of the region of interaction. The difference can be explained from the treatment of the thin airfoil integral in Eq. (3). It will disappear when the domain of integration of this integral is extended somewhat upstream of $x = x_b$. The skin friction for the larger value of the Reynolds number, $Re = 36 \times 10^4$, shows a remarkable dip shortly before reattachment. A similar dip has been observed by Briley and McDonald[21] in Navier-Stokes solutions.

The best impression of the influence of the interaction between the viscous and inviscid parts of the flow can be obtained from Fig. 7. Here we give results for the pressure $p = \frac{1}{2}(1 - u_e^2)$ obtained with interaction. Also presented is the pressure without interaction $\frac{1}{2}(1 - u_{e_0}^2)$. The difference between the two curves indicates the influence of the interaction. Further, the pressure distribution for the larger value of Re possesses a typical plateau region as has been predicted by Stewartson and Williams[29] for supersonic, self-induced separation. Also in experiments such a plateau region can be observed, see Ref. 30 for instance. Finally Fig. 8 presents some velocity profiles for the separated boundary layer with $Re = 36 \times 10^4$. It is noted that the largest backflow velocities, which have a magnitude of about 10% of the edge velocity, occur just before the reattachment point.

Trailing-Edge Flow

As a second example we have used the quasi-simultaneous method to calculate the interaction near the trailing edge of a flat plate placed parallel to a uniform oncoming flow. The Reynolds number of the problem (based upon the length of the plate) is $Re = 10^5$. The plate occupies the interval [0,1] of the x axis. We have assumed that the interaction is limited to the interval of $0.5 \leq x \leq 1.5$. Between $x = 0$ and 0.5 we let the flow be described by the Blasius formulas. In view of the rapid decrease of the displacement thickness in the wake, the thickness $H(x)$ of the computational domain has been chosen equal to $7\delta_{Bl}(x)$, where δ_{Bl} is the Blasius displacement thickness. In this case, therefore, $H(x)$ is constant throughout the global iteration process.

Some results obtained with a 121 × 61 grid (which requires 11 global iterations) have been presented in Figs. 9-11. Here we show displacement thickness, pressure, skin friction, and velocity at the wake centerline. These results have been compared with asymptotically obtained composite expansions formed from the solution of the triple-deck equations,[14] the Blasius values, and the solution of the classical wake

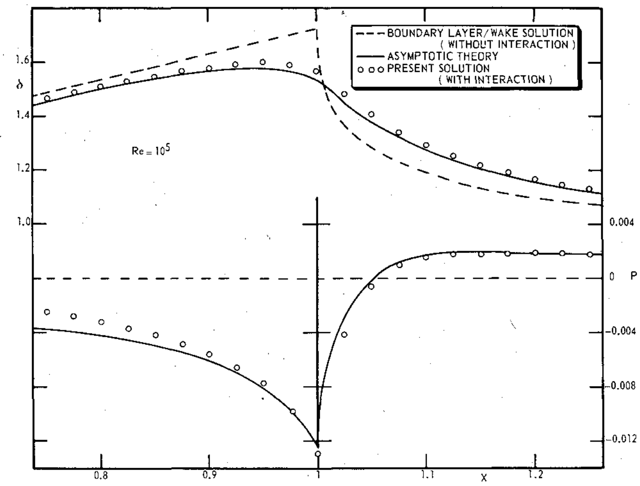

Fig. 9 Trailing-edge flow: comparison of displacement thickness and pressure from present interacting boundary-layer solution, classical boundary-layer/wake solution, and composite solution from asymptotic triple-deck theory.

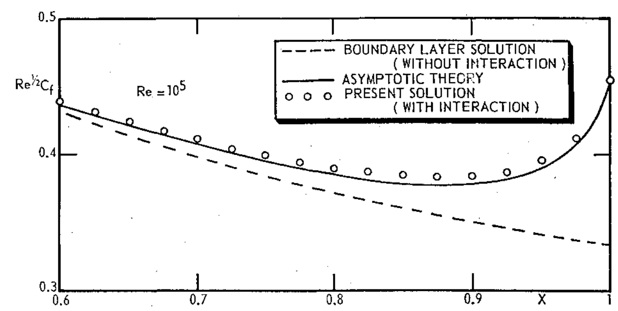

Fig. 10 Trailing-edge flow: comparison of skin friction (see caption of Fig. 9).

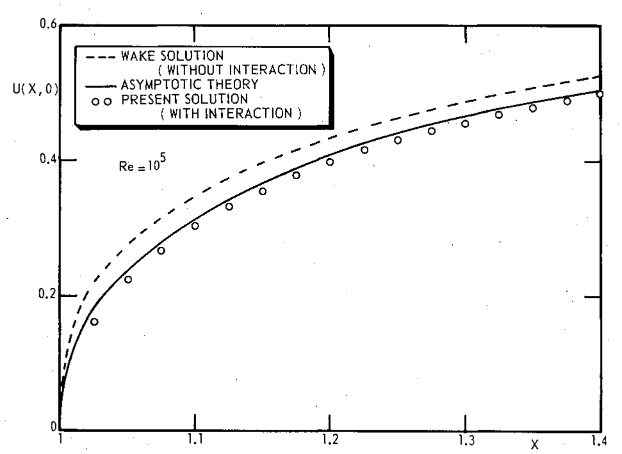

Fig. 11 Trailing-edge flow: comparison of velocity on wake centerline (see caption of Fig. 9).

equations.[31] It is observed that the agreement between the two sets of results is very good, which confirms the consistency between the asymptotic triple-deck theory and the present interacting boundary-layer model. A similar comparison for supersonic interaction near a compression corner has been made by Burggraf et al.,[32] whereas a comparison for subsonic interaction of a boundary-layer flow over a small hump has been performed by Ragab and Nayfeh.[33]

In this example of trailing-edge flow another advantage of the interacting boundary-layer model appears. The classical solution for the displacement thickness possesses a discontinuous tangent $d\delta/dx$ at the trailing edge (see Fig. 9). Use of this displacement thickness as an input to an improved potential flow calculation consequently leads to singularities. To overcome these difficulties it is practice to smooth away the singular behavior of the displacement thickness by some more or less arbitrary procedure. However, the present interacting boundary-layer model automatically leads to smooth results for the displacement thickness.

V. Conclusion

Insprired and guided by asymptotic theory a new, quasi-simultaneous method has been developed for the calculation of interacting boundary layers. The essence of the method is a new interactive boundary condition prescribing a linear combination of pressure and displacement thickness. This linear relation models the behavior of the outer potential flow. This way the advantages of the existing direct and inverse methods are combined, avoiding difficulties near separation points and providing a smooth transition from a directly calculated part of the boundary layer. Furthermore the convergence of the global iteration process is very fast; overrelaxation can be applied. These favorable properties can be explained from the quasi-simultaneous treatment of boundary layer and outer flow.

This quasi-simultaneous method is capable of efficiently calculating incompressible, laminar boundary layers with small separation bubbles, and trailing-edge flows. It is our belief that the philosophy behind the present method can also be used to design efficient calculation methods for turbulent and compressible flows, including flows with shock-wave boundary-layer interaction.

References

[1]Brown, S. N. and Stewartson, K., "Laminar Separation," *Annual Review of Fluid Mechanics,* Vol. 1, 1969, pp. 45-72.

[2]Williams, J. C., "Incompressible Boundary-Layer Separation," *Annual Review of Fluid Mechanics,* Vol. 9, 1977, pp. 113-144.

[3]Catherall, D. and Mangler, K. W., "The Integration of the Two-Dimensional Laminar Boundary Layer Equations Past the Point of Vanishing Skin Friction," *Journal of Fluid Mechanics,* Vol. 26, 1966, pp. 163-182.

[4]Carter, J. E., "Solutions for Laminar Boundary Layers with Separation and Reattachment," AIAA Paper 74-583, June 1974.

[5]Klineberg, J. M. and Steger, J. L., "On Laminar Boundary-Layer Separation," AIAA Paper 74-94, Jan. 1974.

[6]Cebeci, T. and Keller, H. B., "Laminary Boundary Layers with Assigned Wall Shear," *Proceedings of 3d International Conference on Numerical Methods in Fluid Mechanics, Lecture Notes in Physics,* Vol. 19, Springer-Verlag, Berlin, 1975, pp. 79-85.

[7]Horton, H. P., "Numerical Investigation of Regular Laminar Boundary-Layer Separation," *Flow Separation,* AGARD-CP-168, 1975.

[8]Klineberg, J. M. and Steger, J. L., "Calculation of Separated Flows at Subsonic and Transonic Speeds, *Proceedings of 3d International Conference on Numerical Methods in Fluid Mechanics, Lecture Notes in Physics,* Vol. 19, Springer-Verlag, Berlin, 1973, pp. 161-168.

[9]Le Balleur, J. C., "Couplage visqueux- non visqueux: méthode numérique et applications aux écoulements bidimensionnels transsoniques et supersoniques," *La Recherche Aérospatiale,* Vol. 183, 1978, pp. 65-76.

[10]Carter, J. E. and Wornom, S. F., "Solutions for Incompressible Separated Boundary Layers Including Viscous-Inviscid Interaction," *Aerodynamic Analysis Requiring Advanced Computers,* NASA SP-347, 1975, pp. 125-150.

[11]Kwon, O. K. and Pletcher, R. H., "Prediction of Incompressible Separated Boundary Layers Including Viscous-Inviscid Interaction," *Journal of Fluids Engineering,* Vol. 101, No. 4, 1979, pp. 466-472.

[12]Carter, J. E., "A New Boundary-Layer Inviscid Iteration Technique for Separated Flow," AIAA Paper 79-1450, July 1979.

[13]Lagerstrom, P. A., "Solutions of the Navier-Stokes Equation at Large Reynolds Number," *SIAM Journal of Applied Mathematics,* Vol. 28, No. 1, 1975, pp. 202-214.

[14]Veldman, A. E. P., "Boundary Layer Flow Past a Finite Flat Plate," Thesis, Mathematics Institute, University of Groningen, The Netherlands, June 1976.

[15]Hayes, W. D. and Probstein, R. F., *Hypersonic Flow Theory,* Academic Press, New York, 1959, Chap. IX.

[16]Lees, L. and Reeves, B. L., "Supersonic Separated and Reattaching Laminar Flows: I. General Theory and Application to Adiabatic Boundary-Layer/Shock-Wave Interactions," *AIAA Journal,* Vol. 2, Nov. 1964, pp. 1907-1920.

[17]Stewartson, K., "Multi-Structured Boundary Layers on Flat Plates and Related Bodies," *Advances in Applied Mechanics,* Vol. 14, 1974, pp. 145-239.

[18]Brown, S. N. and Stewartson, K., "Trailing-Edge Stall," *Journal of Fluid Mechanics,* Vol. 42, No. 2, 1970, pp. 561-584.

[19]Messiter, A. F., "Laminar Separation - A Local Asymptotic Flow Description for Constant Pressure Downstream," *Flow Separation,* AGARD-CP-168, 1975.

[20]Burggraf, O. R., "Asymptotic Theory of Separation and Reattachment of a Laminary Boundary Layer on a Compression Ramp," *Flow Separation,* AGARD-CP-168, 1975.

[21]Briley, W. R. and McDonald, H., "Numerical Prediction of Incompressible Separation Bubbles," *Journal of Fluid Mechanics,* Vol. 69, 1975, pp. 631-656.

[22]Crimi, P. and Reeves, B. L., "Analysis of Leading-Edge Separation Bubbles on Airfoils," *AIAA Journal,* Vol. 14, Nov. 1976, pp. 1548-1555.

[23]Veldman, A. E. P. and Dijkstra, D., "A Fast Method To Solve Incompressible Boundary-Layer Interaction Problems," *Proceedings of the 7th International Conference on Numerical Methods in Fluid Dynamics,* Stanford, Calif., June 1980.

[24]Melnik, R. E., Chow, R., and Mead, H. R., "Theory of Viscous Transonic Flow over Airfoils at High Reynolds Number," AIAA Paper 77-680, June 1977.

[25]Adamson, T. C., Jr., and Feo, A., "Interaction between a Shock-Wave and a Turbulent Boundary Layer in Transonic Flow," *SIAM Journal of Applied Mathematics,* Vol. 29, 1975, pp. 121-145.

[26]Cebeci, T. and Bradshaw, P., *Momentum Transfer in Boundary Layers,* Hemisphere/McGraw Hill, Washington, D. C., 1977.

[27]Reyhner, T. A. and Flügge-Lotz, I., "The Interaction of a Shock Wave with a Laminar Boundary Layer," *International Journal on Non-linear Mechanics,* Vol. 3, No. 2, 1968, pp. 173-199.

[28]Veldman, A. E. P., "A Numerical Method for the Calculation of Laminar, Incompressible Boundary Layers with Strong Viscous-Inviscid Interaction," National Aerospace Laboratory NLR, Amsterdam, Report NLR TR 79023 U, 1979.

[29]Stewartson, K. and Williams, P. G., "On Self-Induced Separation II," *Mathematika,* Vol. 20, 1973, pp. 98-108.

[30]Van Ingen, J. L., "On the Calculation of Laminar Separation Bubbles in Two-Dimensional Incompressible Flow," *Flow Separation,* AGARD-CP-168, 1975.

[31]Veldman, A. E. P., "A New Calculation of the Wake of a Flat Plate," *Journal of Engineering Mathematics,* Vol. 9, 1975, pp. 65-70.

[32]Burggraf, O. R., Rizzetta, D., Werle, M. J., and Vatsa, V. N., "Effect of Reynolds Number on Laminar Separation of a Supersonic Stream," *AIAA Journal,* Vol. 17, April 1979, pp. 336-343.

[33]Ragab, S. A. and Nayfeh, A. H., "A Comparison of the Second-Order Triple-Deck Theory and Interacting Boundary Layers for Incompressible Flows Past a Hump," AIAA Paper 80-0072, Pasadena, Calif., Jan. 1980.